Sustainability and bifurcations of positive attractors

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Abstract

In this paper we show how sustainability can be rigorously defined by making reference to the positivity of the attractors of a dynamical system. Consistently, the sustainability analysis with respect to various system and policy parameters can be performed by using specialized software for the study of the bifurcations of nonlinear dynamical systems. By means of an example concerning the tourism industry, we show how the analysis can be systematically organized and how easy it is to interpret the results of the numerical bifurcation analysis.

1 Introduction

The notion of sustainability is, nowadays, one of the most pervasive (if not invasive) in all political debates. The idea of sustainability emerged in the late sixties and can be traced in some pioneering scientific work, like that of Hardin (1968) or those that inspired the Club of Rome (Forrester, 1971), and of various socio-political movements (`http://greenpeace.org`, `http://wwf.org` and `http://zerogrowth.org` are few among the hundreds).

A great number of studies followed the pioneering stage and gave rise to *conferences*, as the 1992 UN Commission on Environment and Development conference in Rio de Janeiro, *journals* (like International Journal of Sustainable Development, Sustainable Development and World Ecology, Environmental Modeling and Assessment, Environment and Development Economics, Ecological Economics and others), *books* (Clark and Munn, 1987; Costanza, 1991; Wackernagel and Rees, 1995; Dodds, 2000; Starke, 2002), *laws*, as the European Directives 337/85 and 11/97 for the Environmental Impact Assessment or the 42/2001 for the Strategic Environmental Assessment, and *international agreements*, as the still unattended Kyoto protocol (see `http://unfccc.int/cop3/` for details). The main result of this huge effort is that people and governments are now much more sensitive than thirty years ago to the problem of long term survival of the world. However, despite this success, the issue of sustainability is still missing a simple and clear theoretical framework. This is very unfortunate, because in the absence of unified theories and methods of analysis any issue, no matter how important it is, becomes vague and anoising and, in the long run, discourages young scientists from investing their skillness.

Since sustainability refers to the possibility of keeping alive forever all meaningfull social and natural compartments of an evolving system (from towns to continents) it is clear that any *formal* definition of sustainability must refer to the long term behavior of some appropriate dynamical system. Thus, one should *a priori* expect that the analysis of sustainability with respect to the parameters characterizing the system (e.g. latitude, resource availability, population, ...) or its government (e.g. standards on emissions, environmental taxation schemes, subsidies, …) can be performed through the study of the bifurcations of the attractors of a mathematical model mimicking the evolution of the real system. This is, indeed, the thesis of this article, which has the ambitious target of establishing a bridge between an important issue (sustainability) and a basic chapter of modern mathematics (bifurcation analysis of dynamic systems).

The paper is organized as follows. In the next section two components of sustainability, called *profitability* and *compatibility*, are defined with reference to an abstract model of the system. The first component takes into account only the social compartment of the system, while the second is only concerned with the environmental aspects. From these definitions it follows that the study of profitability and compatibility can be carried out through the bifurcation analysis of the attractors of the model. However, not all the attractors of the system are involved, but only those which are "positive" with respect to the social or to the environmental variables. Then, in the third section, sustainability is defined by putting social and environmental aspects at the same level of importance. This definition is in line with the theory of conflict resolution in multiobjective analysis and it is not as partisan as others proposed by many economists and environmentalists. Again, from our definition it follows that a bridge can be established between sustainability and bifurcation theory. Finally, an entire section is devoted to highlight through an example the meaning of the various definitions given in the paper and to show how the bridge established with bifurcation theory can allow one to systematically and effectively discuss sustainability once a model of the system is available.

2 Profitability and compatibility

We now assume that the time evolution of the variables involved in the problem under consideration is described by a set of ordinary differential equations (ODE). We also assume that the state vector can be partitioned in three subvectors x, y and z of dimensions n_x, n_y and n_z, respectively, and that the variables x_i, $i = 1, \ldots, n_x$ and y_j, $j = 1, \ldots, n_y$ are indicators of social and environmental value. For example, the variables x_i could be measures of employment, welfare, health or education in a given nation, while the variables y_j could be abundances of some plant or animal species in a forest, air quality in various towns, water quality in some rivers and lakes and so on.

All these variables x_i and y_j are typically non-negative, because they represent, directly or not, densities or biomasses. The equations describing the evolution of x_i and y_j over time are simply *conservation equations* involving the balance between inputs and outputs. Moreover, the input and output rates in the balance equations are, with almost no exception, expressed in terms of net per capita rates. In other words, the rate of variation of x_i ($dx_i/dt = \dot{x}_i$) is the product of the abundance x_i and the net growth rate per capita f_i, which is a function of all variables. All this brings to the conclusion that the model can be assumed to have the following general form

$$\dot{x}_i = x_i f_i\,(x, y, z, p, q) \qquad i = 1, \ldots, n_x \tag{1}$$

$$\dot{y}_j = y_j g_j\,(x, y, z, p, q) \qquad j = 1, \ldots, n_y \tag{2}$$

$$\dot{z}_k = H_k\,(x, y, z, p, q) \qquad k = 1, \ldots, n_z \tag{3}$$

where f_i and g_j are net growth rates per capita, z_k are variables of no direct social and environmental interest, and p and q are constant parameters which identify the characteristics of the system (altitude, structure of trasportation networks, fleet dimension, ...) and of its management (emission standards, fishing quotas, subsidies for tourism development, ...).

The particular form of eqs. (1,2) (sometimes called Kolmogorov's form) is such that the non-negativity of the variables x_i and y_j is preserved forever if it is guaranteed at the initial time $t = 0$, i.e. the space $x_i \geqslant 0$, $y_j \geqslant 0$ for all i, j is an invariant set. For physical reasons, in the following we will always refer to this invariant set even if we do not say it explicitly. Given the pair (p, q), i.e. given system (1-3), all its attractors in the above invariant set are uniquely identified (even if not always easily computable). These attractors can be many. Some of them can be not positive – i.e. $x_i = 0$ and $y_j = 0$ for some (i, j) –, while others are positive with respect to x (i.e. the attractor is characterized by $x_i > 0\ \forall i$) and/or with respect to y (i.e., $y_j > 0\ \forall j$). From now on, an attractor which is positive with respect to x will be called x-positive, and similarly for y, while an attractor which is positive with respect to x and y will be called (x, y)-positive. The existence of an x-positive attractor guarantees the possibility that all compartments of social interest remain alive forever. From an economic viewpoint, this means that the system has the possibility of producing profits forever. This justifies the following definition.

Definition 1 *A pair (p, q) is profitable if system (1-3) has at least one x-positive attractor.*

From this definition it follows that the points (p^*, q^*) of the boundary of the profitability region in the space (p, q) are bifurcation points of system (1-3). In fact, given a boundary point (p^*, q^*) there exist pairs (p, q) infinitely close to (p^*, q^*) for which system (1-3) has an attractor characterized by $x_i > 0 \quad \forall i$. By varying the parameters p and q, the attractor must cease to exist or to be x-positive at (p^*, q^*). In the first case the attractor has a catastrophic bifurcation at (p^*, q^*); for example, if the attractor is an equilibrium it disappears through a saddle-node bifurcation or through a subcritical Hopf bifurcation, if it is a limit cycle it disappears through a tangent bifurcation of cycles or through a homoclinic bifurcation, and so on for more complex attractors (i.e., tori and strange attractors). In contrast, if the attractor does not disappear at (p^*, q^*) but loses its positivity with respect to x at that point, then system (1-3) has a transcritical bifurcation at (p^*, q^*). This is a consequence of the Kolmogorov's form of eq. (1) which has the constant solution $x_i = 0$ for all values of p and q Notice that transcritical bifurcations are generic in Kolmogorov's systems (Kuznetsov, 1995).

It is worth noticing that not all bifurcations of system (1-3) are involved in performing the profitability analysis of a system. In fact, bifurcations of attractors which are not positive with respect to x have nothing to do with profitability. The same is true for x-positive attractors which remain such while undergoing a non-catastrophic bifurcation (e.g. a supercritical Hopf bifurcation). Finally, if the system has multiple x-positive attractors, a catastrophic or transcritical bifurcation of one of them does not imply the loss of profitability, which is guaranteed by the remaining x-positive attractors.

It is important to notice that the persistence of all social characteristics is not always guaranteed in a profitable system. In fact, in such a system, besides an x-positive attractor $A'(p, q)$, there can also be another attractor $A''(p, q)$ which is not positive with respect to x, namely an attractor characterized by $x_i = 0$ for some i. In such a case, a sufficiently strong perturbation, like a political scandal, an epidemics or a war, can move in a very short time the state of the system from the attractor A' into the basin of attraction of the alternative attractor A''. Thus, after the perturbation has ceased, the system will tend toward A'' and lose some of its social characteristics. It is therefore lecit to distinguish between safe and risky profitability as follows.

Definition 2 *A profitable pair (p, q) is safe if all the attractors of system (1-3) are x-positive, and risky otherwise.*

Analogous considerations hold for the environmental characteristics of the system: when it is possible to preserve them forever, we say that the system is compatible.

Definition 3 *A pair (p, q) is compatible if system (1-3) has at least one y-positive attractor.*

We can also distinguish between safe and risky compatibility as follows.

Definition 4 *A compatible pair (p, q) is safe if all the attractors of system (1-3) are y-positive, and risky otherwise.*

Of course, the boundary of the compatibility region in the space (p, q) enjoyes the same properties pointed out for the boundary of the profitability region. Thus, points (p^*, q^*) belonging to that boundary are catastrophic or transcritical bifurcation points of system (1-3).

3 Sustainability

In order to give a definition of sustainability which is not as partisan as the two previous ones, we pretend that both the socio-economic and the environmental compartments of the system persist forever. In other words, our formal definition of sustainability is the following.

Definition 5 *A pair (p, q) is sustainable if system (1-3) has at least one (x, y)-positive attractor.*

Notice that a sustainable system is both profitable and compatible, while the converse is not always true. In fact, a system could have many attractors, some x-positive and some y-positive, without having one (x, y)-positive attractor.

As far as risk is concerned, in the case of sustainability one can distinguish between economic and environmental risk as follows.

Definition 6 *A sustainable pair (p, q) is safe if all attractors of system (1-3) are (x, y)-positive, and risky otherwise. Moreover, a risky sustainable pair (p, q) is at economic [environmental] risk if one of the attractors of system (1-3) has $x_i = 0$ for some i [$y_j = 0$ for some j].*

Of course, the boundary of the sustainability region in the space (p, q) can be determined through bifurcation analysis by looking, in particular, at the catastrophic and transcritical bifurcations of the (x, y)-positive attractors. Moreover, also the boundaries separating safe and risky systems are composed of bifurcation points.

4 An example

The example we present in this section concerns tourism sustainability. The model is an extension of a simpler model described in detail in Casagrandi and Rinaldi (2002). The problem is rather abstract and refers to a hypothetical site characterized by four variables: number of tourists (x), environmental quality (y), and capital, subdivided into two quotas (z_1 and z_2) measuring the value (amount) of structures producing touristic services of different nature (e.g., cultural and recreational). The model has the following form

$$\dot{x} = x\left[\mu_y \frac{y}{y+\varphi_y} + \mu_z \frac{\lambda z_1 + (1-\lambda)z_2}{\lambda z_1 + (1-\lambda)z_2 + \varphi_z x + \varphi_z} - \alpha x - a\right] \quad (4)$$

$$\dot{y} = y\left[r\left(1-\frac{y}{K}\right) - \beta_1 z_1 - \beta_2 z_2 - \gamma x\right] \quad (5)$$

$$\dot{z}_1 = -\delta z_1, +\nu \varepsilon x \quad (6)$$

$$\dot{z}_2 = -\delta z_2 + (1-\nu)\,\varepsilon x \quad (7)$$

At the righ-hand-side of eq. (4), the first term is the flow of tourists attracted by the environmental quality of the site, while the second is the flow of tourists attracted by services (with λ and $(1-\lambda)$ representing the relative preferences for the two classes of services); the third term is negative and specifies how quickly the tourists abandon the site when it is crowded, while the last term is the basic rate at which tourists abandon the site. If the site is absolutely not attractive $(y = z_1 = z_2 = 0)$ and there is no crowd, the tourists decay as $exp(-at)$. Of course, the rate a of this exponential decay is higher if there are many other interesting touristic sites. It is therefore reasonable to assume, as done in the following, that the parameter a is a measure of the competition of the alternative touristic sites.

In eq. (5) the first term says that in the absence of touristic activities $(x = z_1 = z_2 = 0)$ the environmental quality recovers to a carrying capacity K in accordance with a logistic equation, while the other terms represent the environmental impacts due to the supply of touristic services and to the presence of tourists.

Equations (6) and (7), which are linear, say very simply that the structures needed for producing services would become obsolete at an exponential rate δ if part of the profit (proportional to the number of visiting tourists) would not be reinvested in the service sector.

Assume that we are interested in the sustainability of the tourism industry and we want to discover the impact of the parameters ε (reinvestment of the touristic agents, for simplicity called *investment*) and a (competition of alternative touristic sites, called *competition*). Moreover, suppose that we also like to detect the effect on sustainability of λ and ν, which represent the preference of the tourists and of the agents for the first class of services. For this we must perform a bifurcation analysis of model (4-7) with respect to ε, a, λ, ν. The first step can be the study of the case $\lambda = \nu = 0.5$, in which the two classes of services are not distinguishable. Figure 1 shows the bifurcation diagram in the space (ε, a) for the reference parameter values given in the caption. The diagram has been obtained using specialized software for bifurcation analysis based on continuation techniques (Khibnik et al., 1993; Doedel et al., 1997). The attractors are either equilibria or limit cycles and there are only five types of bifurcations (Kuznetsov, 1995), that is

TC_{eq} = transcritical bifurcation of equilibria;

SN_{eq} = saddle-node bifurcation of equilibria;

$PLASN_{eq}$ = saddle-node bifurcation of equilibria in the plane $y = 0, z_1 = z_2$;

$HOPF$ = supercritical Hopf bifurcation;

HOM = homoclinic bifurcation.

Figure 1 around here

Figure 1 shows that there are two codimension-2 bifurcation points: a Bogdanov-Takens (see point BT, where a saddle-node, a Hopf, and a homoclinic bifurcation curve merge) and a degenerate saddle-node (see, point $TCSN$, where a transcritical and a saddle-node bifurcation curve merge). The space (ε, a) is partitioned into 10 regions, each one characterized by a specific set of attractors. From Definition 1 it follows that the system is not profitable only in region 8, i.e. the tourists can be permanently present on the site if the competition of the alternative sites is not too high. Similarly, from Definition 3 it follows that the system is not compatible only in region 10, i.e. the environment is necessarily jeopardized by tourism activities if agents reinvest a lot (ε high) and alternative sites are not very competitive (a low). Finally, from Definition 5 it follows that the system is not sustainable in regions 8, 9, and 10, i.e. not only where it is not profitable (region 8) and where it is not compatible (region 10), but also in region 9 where the system is both profitable and compatible.

Figure 1 explicitly shows that not all bifurcations are boundaries of the profitability, compatibility, and sustainability regions. For example, the Hopf bifurcation curve is not a boundary of these regions.

The same bifurcation curves can be used to further partition the profitability, compatibility, and sustainability regions into safe and risky subregions. For example, Figure 2, which has been extracted from Figure 1, shows the subregions in which sustainability is safe and those in which it is at economic and/or environmental risk. From Figure 2 one can immediately conclude that the system can be sustainable and safe only if the alternative sites are not too competitive and the agents are not too aggressive in reinvesting their profits. An increase of competition first gives rise to an economic risk and then to the collapse of the tourism activities. Viceversa, an increase of the aggressivness ε of the agents first generates some environmental risk and finally jeopardizes the environment.

Figure 2 around here

Once the analysis for the symmetric case $\lambda = \nu = 0.5$ has been performed, the parameters λ and ν can be relaxed and the same software can be used to complete the analysis through continuation, starting from Figs. 1 and 2. Two bifurcation diagrams produced in this way are shown in Fig. 3. The first refers to the case in which tourists are more interested in the first kind of services $(\lambda = 0.8)$, while agents invest primarily in the second class of services $(\nu = 0.2)$. The effect, with respect to the symmetric case, is an increase of the safe sustainability region and of that at environmental risk. In the opposite case, i.e. when agents adapt to the preferences of the tourists $(\lambda = \nu = 0.2)$, the above regions shrink and the system becomes more robust with respect to the competition of the alternative sites. All these properties, as well as others that could be easily obtained through continuation, are extremely useful for deriving qualitative but meaningful conclusions on tourism sustainability.

Figure 3 around here

5 Conclusions

Defining sustainability is a very difficult task, since everyone has different perspectives on what should be sustained. In this article we have tried to look at two of the most commonly visited viewpoints. According to the majority of socio-economical scientists, humans well-being and prosperity are issues of primary importance and should be saved with the highest priority. For them, Nature is often the lower trophic level at the expenses of which we can happily survive and grow. Other researchers, especially conservation biologists and philosophers, claim that we are but a very marvellous species that must coexist with other natural beauties, like animals and plants, clear lakes, pristine seas or green mountains with pure air. Many human activities often sounds to them like disturbances that interfere with the organic life of the biosphere. Mediation between these viewpoints is impossible. However, everyone of us has some economic interests in her/his everyday life *and* has the profound necessity of interacting with an alive environment, not simply for exploiting it in the present or in the near future.

We decided to be extreme here. Compatible is every policy that does not completely destroy the environment. Profitable is every policy that ensures some persistent income, no matter how little it can be. Evidently no conservation biologist or economist would easily accept these crude definitions, but surely they will agree that a non-compatible or non-profitable policy cannot be *realistically* sustained in the long run. If the policy is such that a situation which is simultaneously compatible and profitable does exist, thus we can hope to sustain a community. Of course, the income can be unaccettably too little or the environmental conditions can be too contaminated.

This is matter of specific considerations or personal judgement and it is hard to see how a formal method applicable in general to any dynamical model can solve the issue. The method we have proposed, however, helps in selecting the set of sustainable policies in a rigourous way. Trashing the unsustainable can be a good starting point, indeed, in extremely complex situations.

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Figure legends

Figure 1 Bifurcation diagram of model (4-7) in the parameter space (ε, a) for the case $\lambda = \nu = 0.5$. The attractors in each region of the parameter space are sketched in the three dimensional space (x, y, $z = z_1 + z_2$). Other parameter values are as follows: $r = K = \alpha = \gamma = \varphi_z = 1$, $\mu_y = \mu_z = 10$, $\varphi_y = \beta = 0.5$, $\delta = 0.1$.

Figure 2 Sustainability diagram of model (4-7) with respect to investment and competition. Parameter values as in Figure 1.

Figure 3 Sustainability diagram of model (4-7) with respect to investment and competition in the asymetric cases where $\lambda = 0.8$ and $\nu = 0.2$ (left panel) or $\lambda = \nu = 0.2$ (right panel). Other parameter values as in Figure 1.

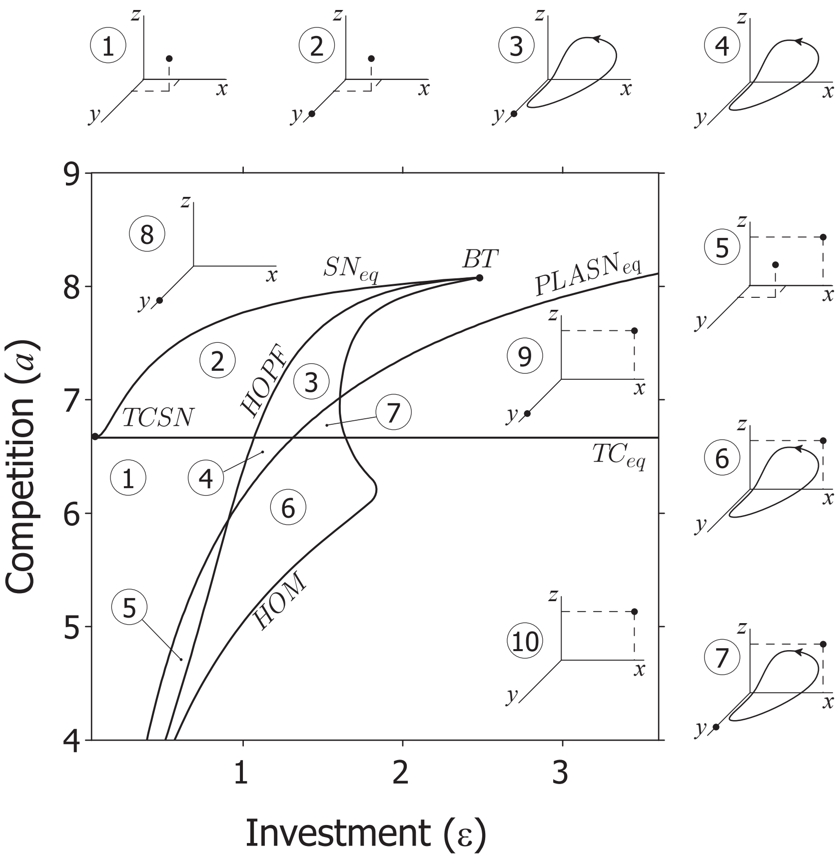

Figure 1 - Casagrandi and Rinaldi

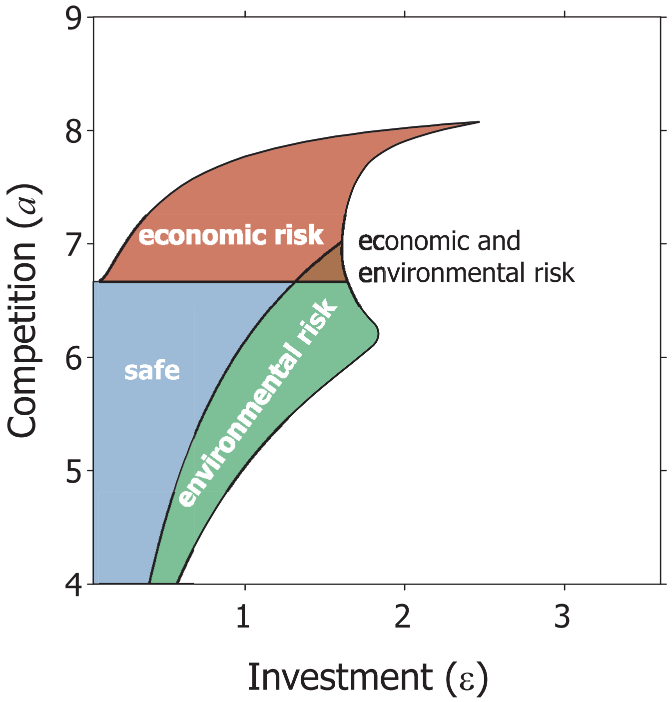

Figure 2 - Casagrandi and Rinaldi

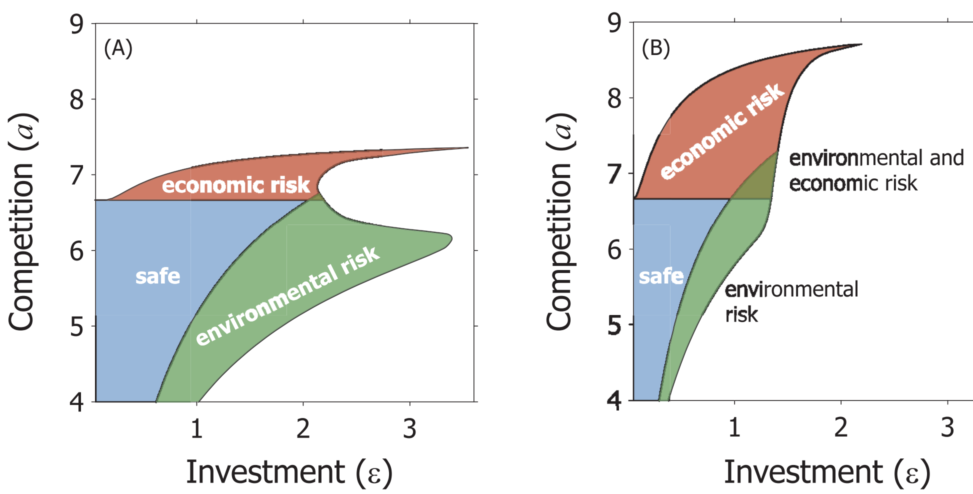

Figure 3 - Casagrandi and Rinaldi